JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **30**, No. 1, February 2017
http://dx.doi.org/10.14403/jcms.2017.30.1.53

SOME CHARACTERIZATIONS OF DEDEKIND MODULES

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Abstract. In this article, we generalize the concepts of several classes of domains (which are related to a Dedekind domain) to a torsion-free module and it is shown that for a faithful multiplication module over an integral domain, we characterize Dedekind modules, cyclic submodule modules, and discrete valuation modules in terms of factorable modules and a sort of Euclidean algorithm.

1. Introduction

Naoum and Al-Alwan, in [9], introduced invertibility of submodules generalizing the concept for ideals and gave several properties and examples of such modules. They also introduced Dedekind and Prüfer modules (An R-module is called a *Dedekind* (resp, *Prüfer*) module if every nonzero (resp., nonzero finitely generated) submodule of M is invertible) and proved that for a faithful multiplication module M over an integral domain R, M is a Dedekind (resp., Prüfer) module if and only if R is a Dedekind (resp., Prüfer) domain. Dedekind modules were further studied in [2, 10, 12]. In [1], Ali gave several properties of invertible submodules of multiplication modules and characterizations of faithful multiplication Dedekind and Prüfer modules.

In this article, we generalize the concepts of several classes of domains (which are related to a Dedekind domain) to a torsion-free module and it is shown that for a faithful multiplication module M over an integral

Received September 30, 2016; Accepted December 16, 2016.
2010 Mathematics Subject Classification: Primary 13A15, 13C13.
Key words and phrases: Dedekind module, discrete valuation module, cyclic submodule module, factorable module, multiplication module.
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This research is financially supported by Changwon National University in 2015-2016.

domain R, we characterize Dedekind modules, cyclic submodule modules, and discrete valuation modules in terms of factorable modules and a sort of Euclidean algorithm.

Let R be an integral domain with quotient field K and M be an R-module. Then M is said to be *faithful* if $ann_R(M) = 0$; M is called a *multiplication module* if each submodule N of M has the form IM for some ideal I of R. Equivalently, M is a multiplication module if and only if for all submodules N of M, $N = (N :_R M)M$. Now M is called a *cancellation module* if for all ideals I and J of R, $IM \subseteq JM$ implies $I \subseteq J$. It was shown in [4, Theorem 3.1] that if R is an integral domain and M is a faithful multiplication R-module, then M is finitely generated. Thus it follows from [4, Theorem 3.1] that a faithful multiplication module M over an integral domain is a cancellation module; thus we have that $I(N :_R M) = (IN : M)$ for all submodules N of M and all ideals I of R. It was also shown in [5, Lemma 2.1] that if M is a faithful multiplication R-module over an integral domain R, then M is torsion-free.

For unexplained terminology and notation, we refer to [6].

2. Factorable modules

In order to characterize Dedekind modules, cyclic submodule modules, and discrete valuation modules in terms of factorable modules, we need the following concept of nonfactorable submodules.

Definition 2.1. Let M be a module over an integral domain R. A submodule N of M is said to be *nonfactorable* if it is a proper submodule and $N = IL$, where I is an ideal of R and L is a submodule of M, implies that either $I = R$ or $L = M$.

When $M = R$ in Definition 2.1, a nonfactorable submodule is called a nonfactorable ideal. Let M be a faithful multiplication module over an integral domain R. If N is a nonfactorable submodule of M, then $(N : M)$ is a nonfactorable ideal of R.

Recall from [1, p.27] that a submodule N of an R-module M is said to be *indecomposable* if $N = IL$, where I is an ideal of R and L is a submodule of M, implies that either $N = L$ or $N = IM$. It is shown that prime submodules are indecomposable and the converse is also true if M is a faithful multiplication Dedekind module over an integral domain R ([1, Corollary 3.5]). It is clear that every nonfactorable submodule is indecomposable, but the converse is not true in general. Also note that neither prime submodules are nonfactorable, nor nonfactorable submodules are prime in general. If M is a faithful multiplication Dedekind module over an integral domain R, then all three concepts mentioned just above are equivalent.

LEMMA 2.2. *Let M be a faithful multiplication module over an integral domain R.*

(1) *If $\mathfrak{n}$ is a nonzero proper ideal of R, then $\mathfrak{n}$ is a nonfactorable ideal of R if and only if $\mathfrak{n}M$ is a nonfactorable submodule of M.*
(2) *If N is a nonzero proper submodule of M, then N is a nonfactorable submodule of M if and only if $(N : M)$ is a nonfactorable ideal of R.*

Proof. (1) Suppose that $\mathfrak{n}M = IL$ for an ideal I of R and a submodule L of M. Since M is a multiplication module, $L = JM$ for some ideal J of R. Then $\mathfrak{n}M = IJM$. Since M is a cancellation module, $\mathfrak{n} = IJ$. Since $\mathfrak{n}$ is a nonfactorable ideal of R, either $I = R$ or $J = R$. In the latter case, we have $L = M$. Therefore $\mathfrak{n}M$ is a nonfactorable submodule of M.

Conversely suppose that $\mathfrak{n}M$ is a nonfactorable submodule of M. Assume that $\mathfrak{n} = IJ$ for some ideals I, J of R. Then $\mathfrak{n}M = I(JM)$. Since $\mathfrak{n}M$ is nonfactorable, either $I = R$ or $JM = M$. In the latter case, we have $J = R$ since M is a cancellation module. Thus $\mathfrak{n}$ is a nonfactorable ideal of R.

(2) Since M is a multiplication module, we have $N = (N : M)M$. N is nonzero proper, so is $(N : M)$. Hence the result follows from (1). □

DEFINITION 2.3. Let M be a module over an integral domain R. Then M is called a *factorable module* if each proper submodule A is a finite product of nonfactorable ideals and a nonfactorable submodule, i.e., $A = I_1 I_2 \cdots I_n N$, where I_i is a nonfactorable ideal of R and N is a nonfactorable submodule of M. Also, M is called a *unique factorable module* if each proper submodule can be factored uniquely into a product of nonfactorable ideals and a nonfactorable submodule.

When $M = R$ in Definition 2.3, a (unique) factorable module is called a (unique) factorable domain.

THEOREM 2.4. *Let M be a faithful multiplication module over an integral domain R. Then M is a (unique) factorable module if and only if R is a (unique) factorable domain.*

Proof. Suppose that M is a (unique) factorable module and let I be a nonzero proper ideal of R. Then IM is a nonzero proper submodule of M. Hence IM can be factored (uniquely) into a product of nonfactorable ideals of R and a nonfactorable submodule of M, say $IM = \mathfrak{n}_1 \cdots \mathfrak{n}_k N$, where each $\mathfrak{n}_i$ is a nonfactorable ideal of R and N is a nonfactorable submodule of M. Thus $IM = \mathfrak{n}_1 \cdots \mathfrak{n}_k (N : M)M$. Since M is a cancellation module, I can be factored (uniquely) into a product of nonfactorable ideals: $I = \mathfrak{n}_1 \cdots \mathfrak{n}_k (N : M)$. Therefore R is a (unique) factorable domain.

Conversely suppose that R is a (unique) factorable domain and let L be a nonzero proper submodule of M. Then there exists a unique nonzero proper ideal I of R such that $L = IM$. By the hypothesis, I can be factored (uniquely) into a product of nonfactorable ideals: $I = \mathfrak{n}_1 \cdots \mathfrak{n}_k$ for some integer $k \geq 1$. Then $L = \mathfrak{n}_1 \cdots \mathfrak{n}_k M = \mathfrak{n}_1 \cdots \mathfrak{n}_{k-1}(\mathfrak{n}_k M)$. Note that $(\mathfrak{n}_k M)$ is a nonfactorable submodule of M. Thus L can be factored (uniquely) into a product of nonfactorable ideals of R and a nonfactorable submodule of M. Therefore M is a (unique) factorable module. □

Compare the following result with [1, Theorem 3.4].

Theorem 2.5. *Let M be a faithful multiplication module over an integral domain R. Then the following statements are equivalent.*

(1) *M is a Dedekind module.*
(2) *M is a unique factorable module.*
(3) *M is a Prüfer factorable module.*
(4) *Every nonzero proper submodule of M can be uniquely expressed as a finite product of nonfactorable submodules.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). By [3, Theorem 5], [9, Theorem 3.4 and Theorem 3.5] (or [12, Theorem 5 and Theorem 8]), Theorem 2.4, and [9, Theorem 3.6].

(1) $\Rightarrow$ (4). Let N be a nonzero proper submodule of M. Since M is a multiplication R-module, there is a nonzero proper ideal I of R such that $N = IM$. Since R is a Dedekind domain, $I = \mathfrak{n}_1 \cdots \mathfrak{n}_k$ (uniquely) for some nonfactorable ideals $\mathfrak{n}_i$ of R. Thus

$$N = IM = \mathfrak{n}_1 \cdots \mathfrak{n}_k M = (\mathfrak{n}_1 M) \cdots (\mathfrak{n}_k M) \text{ (uniquely)}.$$

By Lemma 2.2(1) each $\mathfrak{n}_i M$ is a nonfactorable submodule of M.

(4) $\Rightarrow$ (1). Let I be a nonzero proper ideal of R. Since M is a cancellation module, IM is a nonzero proper submodule of M. Thus $IM = N_1 \cdots N_k$ (uniquely) for some nonfactorable submodules of M.

By Lemma 2.2(2) each $\mathfrak{n}_i := (N_i : M)$ is a nonfactorable ideal of R. Since M is a multiplication module, $N_i = (N_i : M)M$. Thus

$$IM = (N_1 : M) \cdots (N_k : M)M.$$

Since M is a cancellation module, $I = \mathfrak{n}_1 \cdots \mathfrak{n}_k$ (uniquely). Thus R is a Dedekind domain, and so M is a Dedekind module. $\square$

Following [1], an R-module M is called a *cyclic submodule module* (CSM) if every submodule of M is cyclic. Obviously a CSM is a Bézout module. It is also shown that a faithful multiplication module over an integral domain R is a CSM if and only if R is a PID. Recall that an R-module M is called a *discrete valuation module* (DVM) if it is a Noetherian valuation module [8]. Compare the following result with [3, Corollary 6].

Corollary 2.6. *Let M be a faithful multiplication module over an integral domain R. Then M is a CSM (resp., DVM) if and only if M is a factorable Bézout (resp., valuation) module.*

Proof. Assume that M is a CSM. Then by [1, Corollary 3.6] M is a Dedekind module. By Theorem 2.5, M is a factorable Bézout module.

Conversely, suppose that M is a factorable Bézout module. By Theorem 2.5, M is a Dedekind module. Since M is a Bézout module, it follows from [1, Corollary 2.4] that M is a GCD module. Hence by [1, Corollary 3.6] M is a CSM.

Assume that M is a DVM. Then M is a Noetherian valuation module. Note that valuation module $\Rightarrow$ Bézout module $\Rightarrow$ Prüfer module. Then by [1, Thorem 3.4] M is a Dedekind module. By Theorem 2.5 again M is a factorable valuation module.

Conversely, suppose that M is a factorable valuation module. Then R is a factorable valuation domain by Theorem 2.4 and [1, Proposition 2.2]. Thus R is a DVR by [3, Corollary 6], that is, R is a Noetherian valuation domain. Thus M is a Noetherian valuation module, i.e., M is a DVM. $\square$

3. Euclidean-like characterizations

Let $\mathscr{S}(M)^+$ denote the set of all nonzero submodules of a multiplication module M. Then $\mathscr{S}(M)^+$ forms a semigroup with identity M under submodule multiplication. For the definition of submodule multiplication, see [7, pp. 572–573].

THEOREM 3.1. *Let M be a faithful multiplication module over an integral domain R. Then M is a CSM (resp., Dedekind module) if and only if there exists a (length) function $l : \mathscr{S}(M)^+ \to \mathbb{N} \cup \{0\}$ such that both of the following conditions hold.*

(a) *If $A \subseteq B$, then $l(A) \geq l(B)$ with $l(A) = l(B)$ if and only if $A = B$.*
(b) *If A and B are nonzero submodules of M such that $A \nsubseteq B$ and $B \nsubseteq A$, then there exist submodules I, J, K of M such that $K = AI + BJ$, K is cyclic (resp., invertible), and $l(K) < \min\{l(A), l(B)\}$.*

Proof. If M is a CSM (resp., Dedekind module), then let $l(L)$ be the number of nonfactorable submodules in a factorization of L. This number is unique by Theorem 2.5, and so l is well-defined. Then condition (a) is immediate. Condition (b) is achieved by letting $K := A + B$ and $I = J := M$.

Conversely, if l is given and A is a nonzero submodule of M, then let a cyclic (resp., an invertible) submodule B of A be chosen such that $l(B)$ is a minimum. We show that $A = B$. Suppose that B is a *proper* submodule of A and let $\alpha \in A \setminus B$. Therefore, by condition (a), $B \nsubseteq R\alpha$ since $l(B) \leq l(R\alpha)$ by the minimality. Thus, by condition (b), there exist submodules I, J, K of M such that $K = IB + J\alpha (\subseteq A)$, K is cyclic (resp., invertible), and $l(K) < \min\{l(B), l(R\alpha)\} = l(B)$, a contradiction. Hence, $A = B$ is cyclic (resp., invertible). $\square$

Acknowledgements

We would like to thank anonymous reviewers for their useful comments on the paper.

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